Aspects of Spintronics

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Abstract In this chapter, transport properties raised by antisymmetric spin-orbit interactions in noncentrosymmetric systems are discussed. We consider magnetoelectric effects, the anomalous Hall effect, the spin Hall effect, and topological transport phenomena which are in analogy with the quantum spin Hall effect realized in Z_2 topological insulators. These topics are supposed to be relevant to potential applications to spintronics.

1 Introduction

Spin-orbit (SO) interactions in electron systems generally induce the coupling between charge degrees of freedom and spin degrees of freedom, giving rise to distinct transport phenomena involving both charge and spin of electrons. A well-known example is the anomalous Hall effect for which a charge Hall current is raised not by the Lorentz force, but by the coupling between a momentum of an electron and a spin moment through the SO interaction.[1, 2, 3, 4, 5] A closely related phenomenon also caused by the SO interactions is the spin Hall effect: a spin Hall current is generated by an applied longitudinal electric field in the absence of a magnetic field.[6, 7, 8, 9] The spin Hall effect opens a possibility of manipulating electron spins coherently, and may be utilized for potential applications to spintronics devices. In systems with noncentrosymmetric crystal structures, in addition to spherical SO interactions, there is an antisymmetric SO interaction,

$$\mathcal{H}_{\mathrm{SO}} = \alpha(\boldsymbol{k} \times \nabla V) \cdot \boldsymbol{\sigma}. \tag{1}$$

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Here ∇V is an asymmetric potential gradient due to atoms, the locations of which break inversion symmetry. The antisymmetric SO interaction (1) introduces another nontrivial coupling between charge degrees of freedom and spin degrees of freedom, which associates parity-violation in momentum space with broken spin-rotational symmetry. This leads to unique transport phenomena such as magnetoelectric effects. [10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21] For instance, a magnetic field coupled to electron spins controls charge current dynamics of electrons, and conversely, a charge current flow induces and affects magnetic moment of electron spins, which implies potential applications to spintronics. In this chapter, we overview the present theoretical understanding on these phenomena associated with the SO interaction in noncentrosymmetric systems. In the sections 2, 3, 4, our main concern are focused on bulk transport phenomena. Some of the above-mentioned effects are related to paramagnetic effects, and hence, drastically influenced by electron correlation effects. Furthermore, some noncentrosymmetric superconductors (NCS) discovered so far are heavy fermion systems, which are regarded as strongly correlated electron systems. Thus, we will present discussions about electron correlation effects on these transport phenomena, examining feasibility of experimental observations of them in heavy fermion NCS. In the section 5, we will discuss a transport phenomenon analogous to the quantum spin Hall effect: spin currents carried by edge excitations which appear on open boundaries of systems. This phenomenon has been extensively studied for a certain class of band insulators. We demonstrate that a similar effect also occurs in NCS under a particular circumstance.

2 Model systems

In the following, our argument for the case of normal states is largely based on the Hamiltonian,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{SO}}, \tag{2}$$

$$\mathcal{H}_0 = \sum_{k,\sigma} \varepsilon_k c_k^\dagger c_k + U \sum_i c_{\uparrow i}^\dagger c_{\uparrow i} c_{\downarrow i}^\dagger c_{\downarrow i}, \tag{3}$$

$$\mathcal{H}_{\mathrm{SO}} = \alpha \sum_k c_k^\dagger \boldsymbol{\mathcal{L}}_0(k) \cdot \boldsymbol{\sigma} c_k, \tag{4}$$

where $c_k^\dagger = (c_{\uparrow k}^\dagger, c_{\downarrow k}^\dagger)$ is the two-component spinor field for an electron with spin $\uparrow$, $\downarrow$, and momentum k. $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with σ_ν, $\nu = x, y, z$, the Pauli matrices. $\mathcal{H}_{\mathrm{SO}}$ is an antisymmetric SO interaction with a coupling constant α. $\boldsymbol{\mathcal{L}}_0(k)$ is given by an average of the operator $(\boldsymbol{k} \times \nabla V)$ over Bloch wave functions. For tetragonal lattice structures and small k, $\boldsymbol{\mathcal{L}}_0(k) = (k_y, -k_x, 0)$, which is the Rashba interaction.[22] We also include an onsite Coulomb repulsion U in $\mathcal{H}_0$ to discuss electron correlation effects on transport properties, which may be important for heavy fermion NCS. For the discussion on superconducting states, which is presented in the section 3 and in the section 7, we add the BCS mean field pairing term

$$\mathcal{H}_{\rm BCS} = -\frac{1}{2}\sum_k [\Delta_{\sigma\sigma'}(k) c^\dagger_{\sigma k} c^\dagger_{\sigma' -k} + h.c.] \tag{5}$$

to the Hamiltonian (2). Here the gap function is [11]

$$\Delta(k) = \Delta_s(k) i\sigma_2 + \Delta_t(k) \boldsymbol{\mathcal{L}}_0(k) \cdot \boldsymbol{\sigma} i\sigma_2. \tag{6}$$

The first (second) term of eq.(6) is the superconducting gap for a spin-singlet (spin-triplet) component. The $\boldsymbol{d}$-vector for the spin-triplet component of (6) is chosen soo as to optimize the antisymmetric SO interaction.

3 Magnetoelectric effect

The existence of the antisymmetric SO interaction $\alpha(\boldsymbol{k} \times \nabla V) \cdot \boldsymbol{\sigma}$ yields nontrivial coupling between charge and spin degrees of freedom, giving rise to magnetoelectric effect. Magnetoelectric effects have been studied extensively for multiferroic systems, i.e. insulators. However, our argument here is focused on itinerant electron systems. This effect is possible in both the normal state and the superconducting state. In particular, in the superconducting state, the magnetoelectric effect involves dissipationless supercurrent, and, in fact, is related to static and thermodynamic properties rather than non-equilibrium transport.

3.1 Normal state

The magnetoelectric effect in the normal metal was originally discussed by Levitov et al.[13, 14] We explain this effect in the case of cubic systems without mirror symmetry. When an electric field is applied, the antisymmetric SO interaction generates the magnetization,

$$\boldsymbol{M} = \hat{\Upsilon} \boldsymbol{E}. \tag{7}$$

Here the magnetoelectric-effect coefficient $\hat{\Upsilon}$ is a tensor. For cubic symmetry, $\hat{\Upsilon}$ is a pseudoscalar, $(\hat{\Upsilon})_{\mu\nu} = \Upsilon \delta_{\mu\nu}$ with $\mu, \nu = x, y, z$. As an inverse effect, an AC magnetic field gives rise to the charge current flow,

$$\boldsymbol{J} = -\Upsilon \frac{d\boldsymbol{B}}{dt}. \tag{8}$$

Eqs.(7) and (8) have opposite signs because the entropy generations $dS = (\boldsymbol{J}\cdot\boldsymbol{E} - \boldsymbol{M}\cdot\dot{\boldsymbol{B}})dt/T$ under these equilibrium processes must be nonzero. Note that the inverse effect (8) involves dissipation due to current flows, and thus requires dynamical magnetic fields which supply the system with energy.

The physical origin of these effects are easily understood as follows. When the charge current flows along the μ-axis ($\mu = x, y, z$), the Fermi surface is deformed into asymmetric shape, and because of the SO interaction which couples momentum of electrons with spins, the deformation of the Fermi surface yields imbalance of distributions of up-spins and down-spins, giving rise to magnetization along the same axis. Conversely, an applied magnetic field in the μ-direction changes the distribution of spins, which also deforms the Fermi surface asymmetrically, leading to the charge current.

It should be noticed that eq.(8) does not include contributions from magnetization current $J^M = c\nabla \times \boldsymbol{M}$. The magnetization $\boldsymbol{M} = \Upsilon E$ is related to the AC magnetic field via $\nabla \times \boldsymbol{E} = -c^{-1}\partial \boldsymbol{B}/\partial t$. Then, we obtain $J^M = c\nabla \times \boldsymbol{M} = -\Upsilon d\boldsymbol{B}/dt$. The total current induced by the AC magnetic field is

$$J_{\rm tot} = J + J^M = -2\Upsilon \frac{d\boldsymbol{B}}{dt}. \tag{9}$$

The charge current is doubled by the magnetization current.

The magnetoelectric effect is possible also for the case of tetragonal systems with the Rashba-type SO interaction. However, in this case, only the off-diagonal components of the magnetoelectric-effect coefficient $\Upsilon_{\mu\nu}$ with $(\mu, \nu) = (x, y)$ or (y, x) are nonzero. Because of broken inversion symmetry along the z-direction, the Onsager relation for $\Upsilon_{\mu\nu}$ is $\Upsilon_{xy} = -\Upsilon_{yx}$. Thus, eqs.(7) and (8) are changed to

$$M_\mu = -\Upsilon_{\mu\nu} E_\nu, \tag{10}$$

$$J_\mu = -\Upsilon_{\mu\nu} \frac{dB_\nu}{dt}. \tag{11}$$

For this definition of $\Upsilon_{\mu\nu}$, the entropy generations dS is nonzero. Since the sign of eq.(10) is negative in contrast to the positive sign in the case of cubic systems (7), the magnetization current partially cancels the magnetoelectric-effect current; i.e. $\boldsymbol{J} + \boldsymbol{J}^M = -c\Upsilon_{xy}(\partial_x E_z, \partial_y E_z, -\partial_x E_x - \partial_y E_y)$. Thus, when these gradients of an electric field are zero, the current induced by the magnetoelectric effect vanishes.[23]

The magnetoelectric-effect coefficient in the normal state $\Upsilon_{\mu\nu}$ is calculated by using the standard linear response theory based on the Kubo formula,

$$\Upsilon_{\mu\nu}(\omega) = \frac{1}{i\omega} K^{\rm ME}_{\mu\nu}(i\omega_n)|_{i\omega_n \to \omega + i0}, \tag{12}$$

$$K^{\mathrm{ME}}_{\mu\nu}(i\omega_n) = \int_0^{1/T} d\tau \langle T_\tau \{S_\mu(\tau) J_\nu(0)\}\rangle e^{i\omega_n \tau}. \tag{13}$$

Here S_μ and J_ν are, respectively, the total spin and the total current defined by

$$S_\mu = \mu_{\mathrm{B}} \sum_k c^\dagger_k \sigma_\mu c_k, \tag{14}$$

$$J_\mu = e \sum_k c^\dagger \hat{v}_{k\mu} c_k, \tag{15}$$

with

$$\hat{v}_{k\mu} = \partial_{k_\mu}(\varepsilon_k + \alpha \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{L}}_0(\boldsymbol{k})). \tag{16}$$

Here we put the g factor equal to 2. In the case that electron-electron interaction is negligible, and mean free path is determined by scattering due to impurities, $\Upsilon_{\mu\nu}$ is easily calculated by using the Green function formalism;

$$K^{\mathrm{ME}}_{\mu\nu}(i\omega_n) = e\mu_{\mathrm{B}} T \sum_{\varepsilon_m} \sum_k \mathrm{tr}[\sigma_\mu \hat{G}(k, \varepsilon_m + \omega_n) \hat{v}_{k\nu} \hat{G}(k, \varepsilon_m)], \tag{17}$$

where the single-electron Green function in the absence of the Coulomb repulsion U is

$$\hat{G}(k, \varepsilon_m) = \sum_{\tau=\pm} \frac{1 + \tau \hat{\boldsymbol{\mathcal{L}}}_0(k) \cdot \boldsymbol{\sigma}}{2} G_\tau(k, \varepsilon_m), \tag{18}$$

$$G_\tau(k, \varepsilon_m) = \frac{1}{i\varepsilon_m - \varepsilon_{k\tau} + i\mathrm{sgn}(\varepsilon_m)\gamma_k}, \tag{19}$$

with $\varepsilon_{k\tau} = \varepsilon_k + \tau\alpha|\boldsymbol{\mathcal{L}}_0(k)|$, $\hat{\boldsymbol{\mathcal{L}}}_0(k) = \boldsymbol{\mathcal{L}}_0(k)/|\boldsymbol{\mathcal{L}}_0(k)|$, and γ_k the quasiparticle damping. ε_n and ω_n are, respectively, fermionic and bosonic Matsubara frequencies. If we assume a spherical Fermi surface, $\Upsilon_{\mu\nu}$ is $\sim e\mu_{\mathrm{B}} m\alpha\ell/v_F$ where ℓ is a mean free path of an electron.

By using more elaborated analysis based on the Fermi liquid theory, we can take account of electron correlation effects on $\Upsilon_{\mu\nu}$, which may be important for heavy fermion NCS. According to this analysis, we obtain a simple relation among $\Upsilon_{\mu\nu}$, the specific heat coefficient γ, and the resistivity ρ:[19]

$$\Upsilon_{\mu\nu} \sim \frac{\mu_{\mathrm{B}}}{e v^*_F \rho} \cdot \frac{\alpha k_F}{E_F} \propto \frac{\gamma}{\rho} \cdot \frac{\alpha k_F}{E_F}. \tag{20}$$

In general, for heavy fermion systems, the resistivity is given by $\rho \sim \rho_0 + AT^2$, with ρ_0 a residual resistivity and A a constant factor $\propto \gamma^2$. At sufficiently low temperatures, for clean systems, $\Upsilon_{\mu\nu}$ can become large.

We now estimate the order of the magnitude of these effects. We assume that the Fermi velocity is $v^*_F \sim 10^5$ cm/s, which corresponds to the mass enhancement of order ~ 100, i.e. a typical value for heavy fermion systems, and that the SO splitting is sufficiently large, e.g. $\alpha k_F/E_F \sim 0.1$. To consider the magnetization induced by an electric field, we assume that the charge current density is $J \sim 1$ A/cm^2. Then, the induced magnetization is estimated as, $M = \Upsilon E \approx \mu_{\rm B}(\alpha k_F/E_F)(J/ev_F^*) \sim 1$ Gauss, which is experimentally measurable. To evaluate the charge current induced by an AC magnetic field we assume that an AC magnetic field $B = B_0 \cos(\omega t)$ with $B_0 \sim 100$ Gauss, and $\omega \sim 100$ kHz is applied, and the normal resistivity is $\rho \sim 10\mu\Omega \cdot$ cm. Then we obtain the charge current, $J = -2\Upsilon(dB/dt) \approx \mu_{\rm B}(\alpha k_F/E_F)(dB/dt)/(ev_F^*\rho) \sim 1$ mA/cm^2. This magnitude is also experimentally accessible. However, in this case, it is required to discriminate between the current due to the magnetoelectric effect and the usual eddy current induced by the time-dependent magnetic field. For cubic systems, the current induced by the magnetoelectric effect is parallel to the direction of the applied magnetic field, and hence perpendicular to the eddy current. These two currents are distinguished by this directional dependence.

3.2 Superconducting state

Magnetoelectric effects in the superconducting state we shall discuss involve equilibrium dissipationless supercurrent in contrast to the non-equilibrium transport in the normal state discussed in the previous section. We shall see that there exist an extra contribution to the supercurrent induced by the Zeeman magnetic field, and conversely, an extra bulk magnetization induced by the supercurrent flow. These phenomena were originally predicted by Levitov et al. and Edelstein.[10, 11, 12, 16, 17, 18, 19, 20, 21] This mentioned supercurrent is an additional contribution to the ordinary one which is due to finite phase gradients. Its physical origin is also the asymmetric deformation of the Fermi surface due to an applied Zeeman magnetic field as in the case of the normal state. However, in the present case of a static Zeeman field, no net current can arise in the normal state because of the cancellation between the contributions due to the changes in the occupation numbers versus the quasiparticle dispersion. A net finite contribution arises only within the superconducting state where this cancellation is no longer perfect. [11, 16] For the realization of this supercurrent flow induced by Zeeman magnetic fields, a system must allow a bulk current flow without dissipation. One example of such a system may be realized by attaching leads made of superconductors to the sample. Without leads to the outside, the current from phase gradient and the magnetoelectric effect must sum to be zero in the ground state and the system must develop instead a finite phase gradient and therefore be in the ”helical state”.[24]

To explain the magnetoelectric effects, we first exploit the Ginzburg-Landau (GL) theory, and later, we will present microscopic analysis. The GL free energy for superconductors without inversion symmetry was derived by Edelstein, Samokhin, and Kaur et al.[15, 26, 24], which reads,

$$
\begin{aligned}
&F_s - F_n = a|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m_\mu}|D_\mu\Psi|^2 \\
&+\frac{\mathcal{K}_{\mu\nu}}{2en_s}B_\mu(\Psi(D_\nu\Psi)^* + \Psi^* D_\nu\Psi) \\
&+\frac{\boldsymbol{B}^2}{8\pi} - \frac{\chi_{\mu\mu}B_\mu^2}{2},
\end{aligned} \tag{21}
$$

where $a = a_0(T - T_{c0})$, $D_\mu = -\hbar\nabla_\mu - 2eA_\mu/c$, A_μ is a vector potential, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, $\boldsymbol{M}$ is a magnetization density, and n_s is a superfluid density. The forth term of eq.(21) with the coefficient $\mathcal{K}_{\mu\nu}$ stems from the antisymmetric SO interaction, and is the origin of the magnetoelectric effects. Differentiating the free energy with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$, we obtain the following relations for the supercurrent density J^s and the magnetization density $\boldsymbol{M}$,[17]

$$
J^s_\mu = J^{\mathrm{dia}}_\mu + \mathcal{K}_{\nu\mu}B_\nu, \tag{22}
$$

$$
M_\mu = -\mathcal{K}_{\mu\nu}\Lambda J^{\mathrm{dia}}_\nu + M^{\mathrm{Zee}}_\mu, \tag{23}
$$

where J^{dia}_μ is the usual diamagnetic supercurrent given by $J^{\mathrm{dia}}_\mu = (\hbar\nabla_\mu\phi - 2eA_\mu/c)/(2e\Lambda)$ with ϕ the phase of the order parameter Ψ, and $\Lambda^{-1} = 4e^2|\Psi|^2/m$. The last term of eq.(23), M^{Zee}_μ, is magnetization due to the usual Zeeman effect. Also, we have put $|\Psi|^2 = n_s$. The second term of the right-hand side of eq.(22) is the supercurrent due to the magnetoelectric effect, and the first term of the right-hand side of (23) is the magnetization induced by the supercurrent flow.

The structure of $\mathcal{K}_{\mu\nu}$ is constrained by the symmetry requirement as described below:

1. *Tetragonal systems with C_{4v} symmetry*— In this case, the systems are invariant with respect to the reflection $x \to -x$ (or $y \to -y$). Under this reflection, the current J_x (J_y) changes its sign, while B_x (B_y) does not. Thus, this symmetry and eq.(22) imply that $-\mathcal{K}_{xx} = \mathcal{K}_{xx} = 0$. (Also, $\mathcal{K}_{yy} = 0$.) Also, under the reflection $x \to -x$, J_z is invariant, while B_z changes its sign. This implies $\mathcal{K}_{zz} = 0$. Furthermore, the systems are invariant under the $\pi/2$-rotation around the z-axis, i.e. $x \to -y$, $y \to x$. This leads to $\mathcal{K}_{xy} \to -\mathcal{K}_{yx} = \mathcal{K}_{xy}$, and $\mathcal{K}_{xz} \to -\mathcal{K}_{yz} = -\mathcal{K}_{xz} = \mathcal{K}_{xz} = 0$. Only $\mathcal{K}_{xy} = -\mathcal{K}_{yx}$ is nonzero. This case is relevant to $\mathrm{CePt_3Si}$ (space group $P4mm$), and $\mathrm{CeRhSi_3}$, $\mathrm{CeIrSi_3}$ ($I4mm$).
2. *Cubic systems without mirror symmetry*— For cubic systems, if we take μ and ν as the principal axes of the crystal structure, $\mathcal{K}_{\mu,\nu} = \mathcal{K}\delta_{\mu\nu}$ holds; i.e. the supercurrent induced by the magnetoelectric effect is

$$
\boldsymbol{J} = \mathcal{K}\boldsymbol{B}. \tag{24}
$$

Since the left-hand side is a polar vector whereas the right-hand side is an axial vector, $\mathcal{K}$ is a pseudoscalar. For cubic systems without mirror symmetry (O symmetry), $\mathcal{K}$ is nonzero. This case is realized in $\mathrm{Li}_2(\mathrm{Pd}_{1-x}\mathrm{Pt}_x)_3\mathrm{B}$ (space group $P4_332$).

3. *Cubic systems with mirror symmetry*— An example of the crystal structure for this case is that with T_d symmetry. The SO interaction is the Dresselhaus type.[25] Eq.(24) is still applicable to this case. However, since the pseudoscalar should vanish in the presence of mirror symmetry, $\mathcal{K}=0$ and thus the magnetoelectric effect is absent.

It is noted that the above lists are by no means exhaustive. The same symmetry constraint is also applicable to the magnetoelectric-effect coefficient in the normal state $\Upsilon_{\mu\nu}$ discussed in the previous section.

In the case of the Rashba SO interaction, the paramagnetic supercurrent induced by Zeeman fields is partially canceled with magnetization current $\boldsymbol{J}^M = c\nabla\times\boldsymbol{M}$. This was first pointed out by Yip in the case of the Rashba interaction. [17] To see this, using eqs.(22), (23), and the relation $\nabla\times\boldsymbol{J}^{\mathrm{dia}} = -\boldsymbol{B}/c\Lambda$, we write down the total current,

$$\begin{aligned}\boldsymbol{J}_s + \boldsymbol{J}_M &= \boldsymbol{J}^{\mathrm{dia}} + c\nabla\times\boldsymbol{M}_{\mathrm{Zee}} \\ &+ c\mathcal{K}\Lambda(-\partial_x J_z^{\mathrm{dia}}, -\partial_y J_z^{\mathrm{dia}}, \partial_x J_x^{\mathrm{dia}} + \partial_y J_y^{\mathrm{dia}}).\end{aligned} \tag{25}$$

The last term of the right-hand side of (25) is the paramagnetic supercurrent. In the complete Meissner state and in the thermodynamic limit, this term vanishes, and thus there is no paramagnetic supercurrent. Yip pointed out that because of this cancellation, the penetration depth is symmetric under the transformation $z\to -z$.[17] However, in finite systems, or in the mixed state, the last term of (25) gives nonzero contributions to the magnetoelectric effect.

In the case of cubic systems without mirror symmetry, the cancellation between the paramagnetic supercurrent and the magnetization current does not occur, and instead, these currents contribute additively for the magnetoelectric effect, as elucidated in [20, 21]. This is due to the fact that the magnetoelectric-effect coefficient is a pseudo scalar $\mathcal{K}_{\mu\nu} = \mathcal{K}\delta_{\mu\nu}$, as explained in the previous section for the case of the normal state.

The magnetoelectric effect coefficient $\mathcal{K}_{\mu\nu}$ can be calculated by using a linear response theory as in the case of the normal state. Since the magnetoelectric effect in the superconducting state is a static and thermodynamic phenomenon, the coefficient $\mathcal{K}_{\mu\nu}$ is given by a static correlation function,

$$\mathcal{K}_{\mu\nu} = K_{\mu\nu}^{\mathrm{ME}}(0). \tag{26}$$

Here the expression of $K_{\mu\nu}^{\mathrm{ME}}$ is the same as eq.(13), but is evaluated in the superconducting state. For the case without electron-electron interaction, $\mathcal{K}_{\mu\nu}$ is calculated from

$$\mathcal{K}_{\mu\nu} = -e\mu_{\rm B}T\sum_{n,k}\frac{1}{2}{\rm tr}[\hat{S}_\mu\hat{\mathcal{G}}(k,\varepsilon_n)\hat{V}_{k\nu}\hat{\mathcal{G}}(k,\varepsilon_n)], \tag{27}$$

where

$$\hat{S}_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu^t \end{pmatrix}, \tag{28}$$

$$\hat{V}_{k\nu} = \begin{pmatrix} \hat{v}_{k\nu} & 0 \\ 0 & -\hat{v}_{-k\nu}^t \end{pmatrix}, \tag{29}$$

with $\hat{v}_{ky}$ defined by eq.(16), and $\hat{\mathcal{G}}(k,\varepsilon_n)$ is the single-electron Green function for the model (2) with the pairing term (5), defined by

$$\hat{\mathcal{G}}(k,\varepsilon_n) = \begin{pmatrix} \hat{G}_s(k,\varepsilon_n) & \hat{F}(k,\varepsilon_n) \\ \hat{F}^\dagger(k,\varepsilon_n) & -\hat{G}_s^t(-k,-\varepsilon_n) \end{pmatrix}, \tag{30}$$

with

$$\hat{G}_s(k,\varepsilon_n) = \sum_{\tau=\pm1}\frac{1+\tau\hat{\boldsymbol{\mathcal{L}}}_0(k)\cdot\boldsymbol{\sigma}}{2}G_{s\tau}(k,\varepsilon_n), \tag{31}$$

$$\hat{F}(k,\varepsilon_n) = \sum_{\tau=\pm1}\frac{1+\tau\hat{\boldsymbol{\mathcal{L}}}_0(k)\cdot\boldsymbol{\sigma}}{2}i\sigma_y F_\tau(k,\varepsilon_n), \tag{32}$$

$$G_{s\tau}(k,\varepsilon_n) = \frac{i\varepsilon+\varepsilon_{k\tau}}{(i\varepsilon+i\gamma_k{\rm sgn}\varepsilon)^2 - E_{k\tau}^2}, \tag{33}$$

$$F_\tau(k,\varepsilon_n) = \frac{\Delta_{k\tau}}{(i\varepsilon+i\gamma_k{\rm sgn}\varepsilon)^2 - E_{k\tau}^2}, \tag{34}$$

Here $E_{k\tau} = \sqrt{\varepsilon_{k\tau}^2+\Delta_{k\tau}^2}$, $\Delta_{k\tau} = \Delta_s(k)+\tau|\boldsymbol{\mathcal{L}}_0(k)|\Delta_t(k)$, and $\Delta_{s(t)}(k)$ is the BCS gap for spin-singlet (spin-triplet) pairs.

Using the Fermi liquid theory, we can take account of electron correlation effects in eq.(26). The most important electron correlation effect appears in the response to a Zeeman magnetic field, i.e. the renormalization of g-factor by effective mass enhancement. In the case with a spherical Fermi surface, up to the first order in $\alpha k_F/E_F$, the magnetoelectric coefficient is simplified as,

$$\mathcal{K}_{\mu\nu} = \frac{e\mu_{\rm B}n_s\alpha}{8\pi^3 zE_F}, \tag{35}$$

where n_s is the superfluid density. $\mathcal{K}_{\mu\nu}$ is amplified by the mass enhancement factor $1/z$. This feature is in contrast to the electron correlation effect on a conventional diamagnetic supercurrent which is suppressed by the factor z. As a result, the magnetoelectric effect in the superconducting state is much more enhanced in heavy fermion systems with large effective mass than in weakly correlated metals. In the derivation of eq.(35), we assumed that there is no strong ferromagnetic spin fluctuations. If the system is in the vicinity of ferromagnetic criticality, there is additional enhancement of $\mathcal{K}_{\mu\nu}$ due to spin fluctuations. the magnetoelectric effect is enhanced by spin fluctuations.

We now discuss the feasibility of experimental observations of these effects. We use material parameters suitable for heavy fermion systems. Then, assuming $\alpha k_F/E_F \sim 0.1$, the electron density $n \sim 10^{22}$ cm^{-3}, the mass enhancement factor $1/z \sim 100$, and $v_s/v_F^* \sim \Delta/E_F \sim 0.01$, we estimate the magnitude of the bulk magnetization induced by the supercurrent as $M \approx \mu_{\rm B} n(\alpha k_F/E_F)(v_s/v_F^*)/(8\pi^3 z) \sim 0.1$ Gauss. The experimental detection of this internal field may be possible. For the above conditions, the magnitude of the paramagnetic supercurrent is also accessible to usual experimental measurements. It should be emphasized again that to detect the paramagnetic supercurrent, one needs to prepare a circuit in which the bulk current flow without dissipation is possible.

4 Anomalous Hall effect

In this section and the next section, we mainly consider transport phenomena in the normal state. The anomalous Hall effect is the Hall effect that is not due to Lorentz force but caused by SO interactions combined with spin polarization raised by an external magnetic field or a spontaneous magnetization in ferromagnets.[1] This effect has been explained in terms of two different mechanisms; (1) intrinsic mechanism due to bulk SO interaction, (2) extrinsic mechanism due to impurity SO scattering. Here, we are concerned with the former effect due to antisymmetric SO interactions. In the case of the Rashba SO interaction, this effect is intuitively understood as follows. When an electric field applied along the y-axis induces the current flow along this direction, deforming the Fermi surface into an asymmetric shape, the distribution of spins, which is constrained to be perpendicular to the Fermi momentum by the Rashba SO interaction, becomes anisotropic. A magnetic field H_z applied along the z-axis gives rise to torque which rotates spins around the z-axis. Because of the anisotropic distribution of spins and the SO interaction, the rotation of spins accompanies the rotation of the asymmetrically deformed Fermi surfaces on the xy-plane. As a consequence, the net current along the x-axis occurs. The Hall current in this situation is carried by electrons with anomalous velocity associated with the SO interaction. The origin of the anomalous velocity is also understood in terms of Berry-phase effects due to the SO interaction [3]; i.e. the modulation of the Bloch wave function $|u_{\boldsymbol{k}}(\boldsymbol{r})\rangle$ due to the SO coupling gives rise to the Berry curvature defined by $\frac{i}{2}\epsilon_{\alpha\beta\gamma}[\langle\frac{\partial u}{\partial k_\alpha}|\frac{\partial u}{\partial k_\beta}\rangle - \langle\frac{\partial u}{\partial k_\beta}|\frac{\partial u}{\partial k_\alpha}\rangle]$, which plays a role similar to a magnetic field, i.e. a curvature of the gauge field, yielding the transverse force on moving electrons. In the case of the Rashba interaction, the anomalous velocity, $\boldsymbol{v}_{\mathrm{A}} = \alpha(\boldsymbol{n}\times\boldsymbol{\sigma})$ with $\boldsymbol{n} = (0,0,1)$, is perpendicular to the z-axis. Thus, the anomalous Hall effect is possible only for magnetic fields along the z-axis.

The anomalous Hall conductivity $\sigma_{xy}^{\mathrm{AHE}}$ can be computed from the Kubo formula for an anomalous-current correlation function. For general forms of SO interactions, the expression for $\sigma_{xy}^{\mathrm{AHE}}$ is quite involved. However, in the case that, for all $\boldsymbol{k}$ on the Fermi surfaces, the SO split of electron bands $\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})|$ is nonzero and sufficiently larger than the magnitude of quasiparticle damping, the expression is much simplified. We also ignore the Coulomb repulsion between electrons, $U = 0$, for simplicity. Then, the anomalous Hall conductivity for the model (2) with the Rashba SO interaction and a magnetic field H_z parallel to the z-axis is given by,[19]

$$\frac{\mathrm{Re}\ \sigma_{xy}^{\mathrm{AHE}}}{H_z} = e^2\mu_{\mathrm{B}}\sum_{\tau=\pm}\sum_{k}\frac{-\tau f(\varepsilon_{k\tau})}{2\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})|^3}\left(\frac{\partial\mathcal{L}_{0x}}{\partial k_x}\frac{\partial\mathcal{L}_{0y}}{\partial k_y} - \frac{\partial\mathcal{L}_{0y}}{\partial k_x}\frac{\partial\mathcal{L}_{0x}}{\partial k_y}\right). \quad (36)$$

In this derivation, we have ignored orbital motions of electrons due to the coupling with a vector potential, which are not important for the anomalous Hall effect. In eq.(36), the quasiparticle damping γ_k does not appear, and thus, the Hall current is dissipationless in the sense that it does not involve any relaxation mechanisms. Eq.(36) is derived assuming $\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})| \gg \gamma_k$. In the case that for a certain $\boldsymbol{k}$, the SO split vanishes, the factor $\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})|$ in the denominator of eq.(36) for this wave number $\boldsymbol{k}$ is replaced with quasiparticle damping γ_k, regularizing possible divergences of (36). In this case, the Hall effect is dissipative in the sense that momentum dissipation mechanisms play an important role.

According to an analysis based on the Fermi liquid theory, in the case with Coulomb repulsion $U \neq 0$, $\sigma_{xy}^{\mathrm{AHE}}$ is enhanced by the mass renormalization factor $1/z$. This is because that the magnetic field H_z couples to the anomalous velocity through the Zeeman effect, and the paramagnetic effect is enhanced by the mass renormalization effect due to electron correlation. More precisely, the enhancement of $\sigma_{xy}^{\mathrm{AHE}}$ due to electron correlation effects is associated to the enhancement of the van-Vleck-like spin susceptibility which is governed by transitions between the SO split bands.[19] For heavy fermion systems, this factor is of the same order as the mass enhancement factor $1/z \approx 100 \sim 1000$, and thus, the anomalous Hall conductivity can be significantly large. For instance, let us assume the resistivity $\rho \sim 10\ \mu\Omega\cdot\mathrm{cm}$, the mass enhancement factor $1/z_{k\tau} \sim 100$, the Fermi velocity $v_F^* \sim 10^5$ cm/s, and the carrier density $n \sim 10^{22}\ \mathrm{cm}^{-3}$. Then, the ratio of $\sigma_{xy}^{\mathrm{AHE}} \sim e^2\mu_{\mathrm{B}}B/(h^2 v_F^* z)$

to the normal Hall conductivity $\sigma_{xy}^{\mathrm{NHE}}$ is estimated as $\sigma_{xy}^{\mathrm{AHE}}/\sigma_{xy}^{NHE} \sim 40$. The anomalous Hall effect overwhelms the normal Hall effect.

An analogous Hall effect for heat current is also possible. The anomalous Hall conductivity for heat current is expressed as,[19]

$$\kappa_{xy}^{\mathrm{AHE}} = \frac{1}{T}(L_{xy}^{(2)} - \sum_{\mu\nu} L_{x\mu}^{(1)} L_{\mu\nu}^{(0)-1} L_{\nu y}^{(1)}), \tag{37}$$

where $L_{\mu\nu}^{(0)}$ is equal to the conductivity tensor $\sigma_{\mu\nu}$, and, in the absence of electron correlation, $U = 0$, for the Rashba model,

$$\frac{L_{xy}^{(m)\mathrm{AHE}}}{H_z} = e^{2-m}\mu_{\mathrm{B}} \sum_{\tau=\pm}\sum_{k} \frac{-\tau(\varepsilon_{k\tau})^m f(\varepsilon_{k\tau})}{2\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})|^3} \left(\frac{\partial \mathcal{L}_{0x}}{\partial k_x}\frac{\partial \mathcal{L}_{0y}}{\partial k_y} - \frac{\partial \mathcal{L}_{0y}}{\partial k_x}\frac{\partial \mathcal{L}_{0x}}{\partial k_y}\right), \tag{38}$$

with $m = 1, 2$. In the case with electron correlation effects, $U \neq 0$, $L_{xy}^{(m)\mathrm{AHE}}$ is enhanced by the mass renormalization factor $1/z$.

It should be noted that in eqs.(36) and (38), electrons away from the Fermi surface give dominant contributions to the anomalous Hall conductivity. This feature is in accordance with the fact that the magnetic response against the magnetic field along the z-axis is governed by the van-Vleck-like term. This observation leads us to an interesting implication for the superconducting state. In the superconducting state, the Hall effect for heat current is possible at finite temperature, and when the superconducting gap is much smaller than the size of the SO splitting, the thermal anomalous Hall conductivity is not affected by the superconducting transition. Furthermore, even in the limit of zero temperature, $\kappa_{xy}^{\mathrm{AHE}}/(TH_z)$ is nonzero, and behaves like in the normal state, even though the quasiparticle density is vanishingly small. The experimental detection of this effect is an intriguing future issue.

5 Spin Hall effect

The SO interaction gives rise to a transverse spin current under an applied longitudinal electric field even in the absence of external magnetic fields. This effect is called the spin Hall effect.[6, 7, 8, 9, 27, 28, 29, 30, 31] The origin of the spin Hall effect is deeply related to the existence of the anomalous Hall effect.[7] To explain this phenomenon, we consider the Rashba model again. Suppose that a longitudinal electric field E_x along the x-direction and a magnetic field H_z along the z-direction are applied to a system, and there is a nonzero anomalous Hall current; i.e. $J^{\mathrm{AHE}}/e = (n_\uparrow v_\uparrow + n_\downarrow v_\downarrow) = \frac{n_\uparrow + n_\downarrow}{2}(v_\uparrow + v_\downarrow) + \frac{n_\uparrow - n_\downarrow}{2}(v_\uparrow - v_\downarrow) \neq 0$ with $n_{\uparrow(\downarrow)}$ density of electrons with up (down) spin and $v_{\uparrow(\downarrow)}$ velocity of electrons with up (down) spin. On the other hand, in the absence of the magnetic field, J^{AHE} must be zero, and also there is no spin magnetization, i.e. $n_\uparrow - n_\downarrow = 0$, which leads to $v_\uparrow + v_\downarrow = 0$. As a result, for $H_z = 0$ and $E_x \neq 0$, there is a nonzero spin Hall current $J^{\rm SHE}/\mu_{\rm B} = n_\uparrow v_\uparrow - n_\downarrow v_\downarrow = \frac{n_\uparrow + n_\downarrow}{2}(v_\uparrow - v_\downarrow) \neq 0$, while the charge Hall current is zero, $J^{\rm AHE} = 0$. From a different point of view, the origin of the spin Hall effect is understood in terms of spin torque raised by the SO interaction.[9] The applied electric field $E_x \neq 0$ changes the x-component of momentum of electrons by $\Delta p_x = eE_x \Delta t$. This raises the change of the SO interaction, $\alpha \boldsymbol{\sigma} \cdot (\Delta p_x \times \nabla V)$. Since the SO interaction can be regarded as an effective Zeeman effect which depends on the direction of momentum, this change gives rise to torque of spins along $\Delta p_x \times \nabla V$. For the Rashba model, the x-component of electron spins for $p_y > 0$ is opposite to that for $p_y < 0$, and thus the spin torque yields the positive (negative) z-component of spins for $p_y > 0$ $(p_y < 0)$, leading to the spin Hall current along the y-direction. Recently, the existence of the spin Hall effect in the Rashba model has been extensively investigated by several authors.[8, 9] For the Rashba model with broken inversion symmetry along the z-axis, the in-plane spin current with the magnetization in the z-direction is considered. Then, the spin Hall conductivity is defined as,

$$\sigma_{xy}^{\rm SHE} = \lim_{\omega \to 0} \frac{1}{i\omega} K^{\rm SHE}(i\omega_n)|_{i\omega_n \to \omega + i0}, \tag{39}$$

$$K^{\rm SHE}(i\omega_n) = \int_0^{1/T} d\tau \langle T_\tau \{ J_x^{sz}(\tau) J_y(0) \} \rangle e^{i\omega_n \tau}. \tag{40}$$

Here the total spin current J_x^{sz} is,

$$J_x^{sz} = \frac{g\mu_{\rm B}}{4} \sum_k c_k^\dagger (\hat{v}_{kx}\sigma^z + \sigma^z \hat{v}_{kx}) c_k, \tag{41}$$

with g the g-factor.

To obtain an explicit formula for the spin Hall conductivity, we assume again that $\alpha|\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})| \gg \gamma_k$ is satisfied for all $\boldsymbol{k}$. Then, in the absence of electron correlation, $U = 0$, a straightforward calculation yields,

$$\sigma_{xy}^{\rm SHE} = \frac{eg\mu_{\rm B}}{2} \sum_k \sum_{\tau=\pm} \frac{\tau f(\varepsilon_{k\tau})}{2\alpha |\boldsymbol{\mathcal{L}}_0(\boldsymbol{k})|^3} v_x \left(\mathcal{L}_{0y}(\boldsymbol{k}) \frac{\partial \mathcal{L}_{0x}}{\partial k_y} - \mathcal{L}_{0x}(\boldsymbol{k}) \frac{\partial \mathcal{L}_{0y}}{\partial k_y} \right), \tag{42}$$

where $v_\mu = \partial_{k_\mu} \varepsilon_k$. As in the case of the anomalous Hall conductivity (36), quasiparticle damping does not appear in the expression (42), which indicates that the effect is dissipationless. In the case of a two-dimensional electron gas model with the Rashba interaction $\boldsymbol{\mathcal{L}}_0 = (k_y, -k_x, 0)$ and $\varepsilon_k = k^2/2m$, when the Fermi level crosses both of two SO split bands, the spin Hall conductivity calculated from (42) is

$$\sigma_{xy}^{\rm SHE} = \frac{eg\mu_{\rm B}}{8\pi}. \tag{43}$$

Remarkably, its value is universal, and independent of any parameters specific to the system such as the SO coupling α and electron density.[9] However, this by no means implies that $\sigma^{\mathrm{SHE}} \neq 0$ even for $\alpha \to 0$. It should be noted that the above result is obtained under the assumption that the SO split is much larger than the quasiparticle damping. As $\alpha \to 0$, the quasiparticle damping which should appear in the denominator of (42) becomes important, leading to $\sigma^{\mathrm{SHE}} \to 0$.[27] In more general cases where the SO interaction is not the Rashba type and the Fermi surface is not spherical, the magnitude of $\sigma_{xy}^{\mathrm{SHE}}$ depends on the detail of the electronic structure and is not universal.

When the quasiparticle damping is governed by impurity scattering, $\sigma_{xy}^{\mathrm{SHE}}$ is partially cancelled with current vertex corrections due to impurity scattering which are related to the single-electron selfenergy (the quasiparticle damping) via the Ward-Takahashi identity. In particular, this cancellation is perfect, $\sigma_{xy}^{\mathrm{SHE}} = 0$, in the case of the Rashba model with $\boldsymbol{\mathcal{L}}_0(\boldsymbol{k}) = (k_y, -k_x, 0)$ even when the SO split is much larger than the scattering rate. However, this complete cancellation is accidental, and does not hold for general forms of SO interactions.[32] Thus, to calculate $\sigma_{xy}^{\mathrm{SHE}}$ correctly, one needs to take account of both the detail band structure and scattering processes which govern the quasiparticle damping.

According to a precise analysis based on the Fermi liquid theory, the spin Hall conductivity $\sigma_{xy}^{\mathrm{SHE}}$ is not affected by electron correlation effects, but determined solely by the band structure, in contrast to the anomalous Hall conductivity discussed before. This is simply due to the absence of paramagnetic effects (Zeeman fields) for the spin Hall effect.[19]

The experimental observations of the spin Hall effect were successfully achieved for semiconductors.[30, 31] In these experiments, spin polarization at the edges of samples due to spin currents under an applied electric field was detected by optical measurements. Unfortunately, experiments for NCS have not been achieved so far, partly because it is difficult, up to now, to synthesize single crystals of NCS with a size large enough to make the detection of the spin Hal effect feasible.

6 Quantum (spin) Hall effect in the superconducting state: topological transport phenomena

The subject in this section is conceptually different from the bulk transport phenomena considered in the previous sections. Here, we discuss transport phenomena raised by nontrivial topological structures of the many-body Hilbert space. As mentioned before, the anomalous Hall effect and the spin Hall effect are also related to a topological property: nonzero Berry curvature in momentum space. However, the topological transport phenomena discussed here are distinct from these Hall effect in that transport currents are carried not by bulk quasiparticles, but by edge excitations which exist on boundaries of systems. Such transport phenomena occur in the case that there are both a bulk excitation energy gap and gapless edge excitations. The studies on topological transport phenomena were initiated in the celebrated paper by Thouless et al., in which the topological explanation for the quantum Hall effect realized in two-dimensional electron gas in a strong magnetic field was presented.[33] In the quantum Hall state, there is a bulk energy gap due to the Landau quantization of the energy band, and Hall currents are mainly carried by gapless edge states, which propagate along one direction only, and are topologically protected from perturbations such as disorder.[34, 35] Here, the topological protection means that the existence of edge states is closely related to a nonzero topological number, i.e. the first Chern number n_{Ch} for the U(1) bundle corresponding to the wave function. That is, the U(1) phase of the wave function is not smooth in the entire (magnetic) Brillouin zone, and there is a jump of the phase somewhere in the $\boldsymbol{k}$-space, which leads to the nonzero Berry curvature, and the nonzero Chern number. As a result, the edge states are stable against any local perturbations which can not change the topology of the Hilbert space. The modulus of the Chern number represents the total number of the edge modes. The Hall conductivity is expressed in terms of the Chern number as $\sigma_{xy} = (e^2/h)n_{\mathrm{Ch}}$.[33]

It was pointed out by several authors that a similar phenomenon is possible in chiral $p + ip$ superconductors, in which there is a gapless edge mode, which propagates along only one direction, reflecting broken time-reversal-symmetry in chiral superconductors.[36, 37]

For a certain class of insulators with time-reversal symmetry, there exists another topological transport phenomenon, which is associated with spin currents, and called the quantum spin Hall effect; i.e. in a certain class of insulators with a bulk energy gap, a spin Hall current is induced by a longitudinal electric field.[38, 39, 40, 41] In this state, the Chern number is zero, because of time-reversal symmetry. However, instead, this state is characterized by another topological number called the Z_2 topological invariant.[38] These insulators are called the Z_2 topological insulators.

As there is the close relation between the quantum Hall state and chiral $p+ip$ superconductors mentioned above, there is also parallelism between Z_2 insulators and $s + p$-wave NCS.[44, 45, 46, 47] Moreover, in the case with a magnetic field, the $s+p$-wave NCS also exhibit a topological phase in analogy with the quantum Hall state characterized by the nonzero Chern number. In the following, we discuss these topological phenomena realized in NCS.

6.1 Z_2 insulator and quantum spin Hall effect

Before considering NCS, we briefly summarize the fundamental properties of the Z_2 insulator relevant to the discussion on NCS. The Z_2 insulator possesses a bulk excitation energy gap which separates the ground state from excited states. In contrast to the quantum Hall state where the bulk gap is due to the filled Landau level, the bulk gap of the Z_2 insulator is a band gap, or a gap generated by some symmetry-breaking of the system which preserves time-reversal symmetry. The most important feature of the Z_2 insulator is the existence of two gapless edge modes which propagate in the opposite directions, and carry, respectively, up-spin and down-spin. This leads to a nonzero spin current flowing on the edge without net charge current flow. As a result of it, the quantum spin Hall effect occurs; i.e. an applied electric field parallel to the edges gives rise to spin Hall current traverse.

The Z_2 insulator is regarded as a pair of two quantum Hall states in which magnetic fields are applied in the opposite directions, and time-reversal symmetry is preserved in the whole system. For a while, we assume that the total spin is conserved. Then, the two quantum Hall states are, respectively, associated with spin up and spin down states. In such a system, each of two quantum Hal states possesses nonzero Chern numbers with the same magnitude but different signs. Thus, the total Chern number is zero. However, there are another topological numbers which characterize the topological phase.[38, 42, 43, 48, 49] Let us consider the case that there are m gapless edge modes ($m > 1$) for each spin state (i.e. the total number of edge modes is $2m$), and that the spin-resolved Chern number (Chern number for each spin state) is m. All of these gapless edge modes are not necessarily topologically protected. For instance, two edge modes in the same spin state may propagate in the opposite directions. In this situation, the two gapless edge modes are backscattered by non-magnetic impurity, and become gapful. Thus, for the case of even m, the system is not topologically-protected. When m is odd, there is, at least, one gapless edge mode which is stable against disorder, characterizing the topological phase. This implies that as long as the topological nature is concerned, there are only two states; i.e. topologically trivial or non-trivial. These two states are classified by the parity of spin-resolved Chern number m. Originally, the topological number which characterizes this topological phase was introduced by Kane and Mele by using the Pfaffian of a matrix $M_{mn}(k) = \langle u_{k,m}|\Theta|u_{k,n}\rangle$ where $|u_{k,n}\rangle$ is the Bloch state with a wave vector k and a band index n ($n = 1, 2, ..., N$), and Θ is the time reversal operator.[38] Note that each Bloch function $|u_{k,n}\rangle$ is two component spinor which consists of the Kramers doublet, and that $M_{mn}(k)$ is a $2N \times 2N$ matrix. The total number ν of zeros of the Pfaffian $\mathrm{Pf}[M(k)]$ in half the Brillouin zone which includes only one of $\boldsymbol{k}$ and $-\boldsymbol{k}$ discriminates between the topological phase and trivial insulators. For the Z_2 topological insulator, $\nu = 1$ (mod 2), and for trivial insulators, $\nu = 0$ (mod 2). Later, it turned out that the Z_2 invariant is equivalent to the parity of the spin-resolved Chern number.[50, 42]

In the above explanation, we consider the case that the spin projection S_z is a good quantum number. However, the concept of the Z_2 invariant is more general and applicable also to the case without spin conservation, as long as time-reversal symmetry is preserved and there is the Kramers degeneracy. Actually, in microscopic models for the Z_2 insulator proposed so far, SO interactions which violate spin conservation play important roles to stabilize the topological phase.[38, 39, 40, 41] When the total spin is not conserved but time reversal symmetry is still preserved, the above argument is valid if we replace the spin-up and spin-down states with the Kramers doublet. The stability of two gapless edge modes which form the Kramers doublet is ensured by the nonzero Z_2 invariant. More precise arguments on the relation between the gapless edge modes and topological numbers, and effects of electron-electron interaction are given in refs.[50, 51].

6.2 Z_2 topological phase in noncentrosymmetric superconductors

As mentioned before, there is parallelism between Z_2 insulators and $s+p$-wave NCS in the absence of magnetic fields.[44, 45, 46, 47] To explain this point, we consider 2D NCS with the Rashba SO interaction defined on a square lattice. We assume the d-vector of the p-wave pairing is compatible with the Rashba interaction, i.e. $\boldsymbol{d} \propto (\sin k_y, -\sin k_x, 0)$. We also allows for the admixture of the s-wave pairing. In two dimension, the superconducting gaps in the two SO split bands have no nodes provided that the p-wave gap $\Delta_p(\boldsymbol{k})$ is not equal to the s-wave gap $\Delta_s(\boldsymbol{k})$ for any $\boldsymbol{k}$ on the Fermi surfaces.. To clarify the topological nature of this system, we consider the energy spectrum of edge states in the case that the geometry of the system is a cylinder with open boundaries at $x = 0$ and $x = L$. [44]. According to the numerical analysis for this system, when $\Delta_p(\boldsymbol{k}) > \Delta_s(\boldsymbol{k})$ is satisfied on the Fermi surfaces, two gapless edge modes on each boundary emerge.[44, 46] The two gapless edge modes on the same boundary are, respectively, associated with the two SO split bands which constitute the Kramers doublet, and propagate in the directions opposite to each other. This state is characterized by the Z_2 topological number, in analogy with the Z_2 insulators.[44, 46] In this phase, each of superconducting states realized in two SO split bands is similar to a chiral $p + ip$ superconducting state with different chirality. Actually, the Hilbert space of this phase can be deformed into a topologically equivalent one which is a product of the spaces of a chiral superconductor with $p_x + ip_y$ gap symmetry and that with $p_x - ip_y$ gap symmetry. The deformation into a topologically equivalent phase is possible when bulk excitation gaps are not closed by this deformation. In the case that $\Delta_p(\boldsymbol{k}) > \Delta_s(\boldsymbol{k})$ is fulfilled on the Fermi surfaces, we are able to change the magnitudes of $\Delta_s(\boldsymbol{k})$ and the SO coupling α continuously to zero without closing the bulk superconducting gap. In this state, because of time-reversal symmetry, the Chern number is zero, and there is no charge Hall current flowing on the edge. However, a spin current carried by the edge states exists, which gives rise to the spin Hall effect, in analogy with the Z_2 insulator.

6.3 Analogue of quantum Hall state in the case with magnetic fields

We consider again the 2D Rashba superconductors with $s+p$-wave pairing gaps satisfying the condition $\Delta_p(\boldsymbol{k}) > \Delta_s(\boldsymbol{k})$ on the Fermi surfaces. In the case with a magnetic field, a topological state similar to the quantum Hall state is realized for a particular electron density.[44] When the Fermi level crosses the Γ point in the Brillouin zone, and a magnetic field is applied to the system, a gap opens at the Γ point. If the magnetic field is smaller than an upper critical field of the superconducting state, one gapless edge mode associated with the band at the Γ point disappears, leaving only one gapless edge mode. This chiral edge state is analogous to the quantum Hall effect state. However, in contrast to the quantum Hall effect state, this gapless edge state does not carry a charge current, because the quasiparticles in the edge state are Majorana fermions; i.e. the antiparticles of them are equivalent to themselves. The Majorana edge state may be probed by thermal transport measurement.

The existence of the gapless edge mode is deeply related to the existence of a zero energy mode in a vortex core which is also described by a Majorana fermion, as clarified by analysing the Bogoliubov-de Gennes equations.[52, 44] In fact, when the geometry of the system is a disk with a closed boundary, and there is no vortex core in the system, i.e. the geometry of the system is simply-connected, the edge mode has an excitation gap of order $1/L$ where L is the perimeter of the closed system. In contrast, when there is a single vortex with odd vorticity in the bulk system, the edge mode becomes gapless, and simultaneously, a zero energy state in the vortex core appears. In this sense, the gapless edge mode is a concomitant of the zero energy vortex core state. A quasiparticle on the edge in the gapless case is also a Majorana fermion. This implies that a Majorana fermion can not exist in isolation, but should always accompany a Majorana partner, with which it forms a complex fermion.

The chiral Majorana edge state is also realizable even in a purely s-wave Rashba superconducor with $\Delta_s \neq 0$ and $\Delta_p = 0$, provided that the Zeeman energy due to a magnetic field H is larger than the s-wave gap Δ_s, i.e. $\mu_{\mathrm{B}}H > \Delta_s$, and that the Fermi level is located within the energy gap around $\boldsymbol{k} \sim 0$ generated by the Zeeman effect.[53, 54] There are several proposals for the realization of this system, which utilize, e.g., ultracold fermionic atoms, heavy fermion superconductors, and semiconductor heterostructures.[53, 54, 55, 56]

6.4 Accidentally protected spin Hall state without time-reversal symmetry

In the case with a magnetic field, because of broken time reversal symmetry, the topological characterization in terms of the Z_2 number is not applicable. However, even in such a situation, a pair of two gapless edge modes which carry a spin current is stable for the 2D Rashba superconductors with the condition $\Delta_p > \Delta_s$, provided that the magnetic field is perpendicular to the direction along which the edge modes propagate.[44] In this phase, both the Z_2 number and the Chern number are zero. However, there is another topological number which ensures the stability of this phase. This topological number is a winding number defined for particular symmetry points in the Brillouin zone inherent in the Rashba model.[44] In this sense, the stability of this phase is accidental, and fragile when there is a magnetic field component parallel to the propagating direction of the edge modes.

6.5 Topological transport phenomena

The transport phenomena associated with edge states can be experimentally detected by using the measurements for a system with a Hall bar geometry as considered before for the case of the quantum Hall effect in two-dimensional semiconductors.[57] In superconducting systems, instead of charge currents in semiconductors, the measurement of a heat current is useful for the detection of quasiparticle contributions to transport phenomena. In the topological phases mentioned above, heat currents are mainly carried by gapless edge states, and hence the thermal conductivity exhibits power law behavior $\propto T$ as a function of temperature, in contrast to the bulk contributions to the thermal conductivity which should decay exponentially at low temperatures $\sim \exp(-\Delta/T)$ in the superconducting state with full gap Δ. A more drastic effect characterizing the existence of edge states is a non-local transport phenomenon. In a Hall-bar geometry in which two terminals (1 and 2) are attached to one of two longer edges and another two terminals (3 and 4) are attached to the other longer edge, the temperature gradient between the terminals 1 and 3 gives rise to a heat current flowing between the terminals 2 and 4. This non-local transport can not be explained if one considers only the contributions from bulk quasiparticles, when the distance between the terminals 1, 3 and the terminals 2, 4 is sufficiently large. The detection of this effect may be a direct evidence for the existence of edge states governing low-energy transport. The experimental verification of these phenomena has not been achieved so far for NCS. The exploration for the topological superconducting state in NCS is an interesting and important future issue.

7 Conclusions

The antisymmetric SO interactions inherent in noncentrosymmetric systems are sources of remarkable transport phenomena both in the superconducting state and in the normal state, which are characterized by nontrivial coupling between charge and spin degrees of freedom. Although experimental verification of these phenomena in noncentrosymmetric superconductors is not yet achieved, it is naturally expected that some of them related to the paramagnetic effect such as the anomalous Hall effect and magnetoelectric effects are enhanced in strongly correlated electron systems, and their experimental detections may be feasible.

The antisymmetric SO interactions are also origins of topological order and topological transport phenomena such as the quantum spin Hall effect. In noncentrosymmetric superconductors under certain circumstances, topological phases akin to Z_2 topological insulators can be realized.

Acknowledgements The work of SF was supported by the Grant-in-Aids for Scientific Research from MEXT of Japan (Grants No.18540347 and No.19052003) The work of SKY was supported by the National Science Council of Taiwan under Grant number NSC95-2112-M-001-054-MY3.

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